

Singular Integral Regularization and Neutrix Subtraction: Analytical and Numerical Approaches

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Received: April 26, 2026 | Review began: May 04, 2026 | Review ended: June 02, 2026 | Published: June 17, 2026

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Abstract

We have all often noticed that a common mathematical challenge recurs in certain areas, which is divergence, whether we are talking about fracture mechanics, quantum field theory, electrostatics, or singular integrals. A very common example is crack mechanics simulations, where the singular integrals of broken steel beams usually diverge, making numerical stability very difficult.

In this paper, we analytically discuss the expansions of multi-dimensional singular integrals using Neutrix calculus, especially in the context of radial singularities. To do this, we isolate finite parts of a broad spectrum of regularization techniques such as Gaussian, rational, exponential, polynomial, Lorentzian, and hyperbolic methods and test their efficiency. The numerical results show that the closer one is to the singular region, the more important the magnitude of the integral can be: in one dimension, on the closed interval from 1 to 10, the value of the integral can vary between 1.845213 and 9.404645; in two dimensions between 9.454510 and 92.057897; in three dimensions between 62.435420 and 82.057897. When using algebraic-type regularizations with a value of $n = 1.2$ and a value of $\epsilon = 0.01$, the raw values of the integrals were similar, with Lorentzian 76.791266, Rational 78.411604, and Polynomial/Hyperbolic 80.304942 representing the values of the algebraic-type regularizations, respectively. However, using Gaussian and Exponential damping, the values of the algebraic-type regularizations were much smaller, with 3.403055 and 4.887252, respectively. The stable finite parts were given by the Lorentzian, Rational, Polynomial, and Hyperbolic methods, but it was found that the Gaussian damping was over-suppressed. Relative errors close to machine precision, in the Lorentzian case, which justify the errors of the numerical implementation, were also indicated by benchmark validation. This paper provides a comprehensive framework, an analytical derivation, and numerical simulations. This research could open new avenues for theoretical physics and computational engineering.

Categories: Signal Processing and Image Analysis, Structural Analysis and Finite Element Analysis, Data Science and Analytics for Engineering

Keywords: generalized functions, neutrix calculus, radial singularities, singular integrals, finite-part extraction

Introduction

Singular and hypersingular integrals are often deeply embedded within potential theory, fracture mechanics, wave propagation, elastodynamics, and Partial Differential Equation-based boundary integral formulations. In the context of boundary element methods, Sladek provided a detailed discussion on the regularization of hypersingular integrals using continuous elements, and also highlighted the importance of carefully handling singular kernels in engineering formulations [1]. Subsequently, Gao developed numerical procedures for 2D singular boundary integrals [2], which were later extended in subsequent works to encompass general higher-order singular boundary integrals in both 2D and 3D

How to cite this article:

Tiwari C, Shukla A, Rastogi S (June 17, 2026) Singular Integral Regularization and Neutrix Subtraction: Analytical and Numerical Approaches. Cureus J Eng 3 : es44388-026-00125-3. DOI https://doi.org/10.7759/s44388-026-00125-3

problems [3]. Significant progress has also been made in recent studies - for instance, Sun analyzed singular boundary integrals within 3D time-domain boundary element methods [4], while Qin explored direct approaches for singular integrals in elastodynamic boundary element formulations [5]. All these studies consistently show that singular kernels are not isolated mathematical anomalies, but rather frequently recurring features of computational physics and engineering models. If we talk about applied mathematics and physics, we will often notice that singular integrals are very common - such as in stress-intensity analysis, Green-function formulations, electromagnetic scattering, boundary element methods, and field propagation problems. The biggest problem is that the kernels of these integrals often diverge (become infinite) when we evaluate near the origin or a lower-dimensional manifold. Simply put, it becomes difficult to obtain finite and stable values from normal integration. Its limitation is that standard integration techniques provide only qualitative ideas, but do not give physically meaningful finite results. That is why regularization and subtraction techniques are required to isolate the divergent (infinite) part and extract the finite, physically relevant contribution.

Several numerical and analytical approaches have been developed for the treatment of singular and hypersingular integral equations. Boykov et al. [6] proposed approximate methods for solving linear and nonlinear hypersingular integral equations, and their later work examined hypersingular integral equations of Prandtl's type together with their numerical methods and applications [7]. Khan et al. [8] studied the computation of Hadamard finite-part integrals with Fourier oscillators, which is especially relevant for oscillatory singular kernels. Chen and Li [9] applied Clenshaw-Curtis quadrature rules for regular and singular integrals in boundary integral equations. Gaussian quadrature techniques for certain two-dimensional hypersingular integrals were studied in [10], while regularized projection methods for weakly singular integral equations were considered in [11]. These works demonstrate that accurate treatment of singular integrals depends strongly on the choice of quadrature, projection, and regularization procedure.

Parallel developments have also appeared in mechanics-based, fractal, oscillatory, and mixed integral-equation settings. Mkhitarian et al. [12] investigated hypersingular integral equations arising in mechanics problems, while Gibbs et al. [13] analyzed singular integrals on non-disjoint self-similar fractal sets. Abusalim et al. [14] developed an algorithm for nonlinear Volterra-Fredholm integral equations with singular kernels, and Alhazmi et al. [15] considered computational techniques for mixed dimensional integral equations with strongly symmetric singular kernels. Oscillatory Fredholm integral equations with weakly singular kernels were studied in [16], and weakly singular nonlinear Volterra integral equations were treated numerically in [17]. Classical finite-part and Neutrix approaches are based on earlier work by Hadamard and van der Corput [18,19]. These contributions confirm that regularization is a structural component of stable computation, not merely a technical correction.

Despite this broad literature, most existing frameworks focus either on numerical quadrature or on problem-specific regularization. Comparatively less attention has been given to a unified Neutrix-based formulation that simultaneously addresses multidimensional radial singularities, finite-part extraction, and cross-method comparison. Neutrix calculus provides a mathematically stable framework in which divergent and finite contributions can be separated systematically without discarding the asymptotic structure of the kernel. Although Neutrix-based and finite-part ideas were originally developed mainly for one-dimensional or distributional settings, their higher-dimensional extensions remain comparatively limited. Numerical integration was performed using adaptive quadrature methods [20-25]. Such singular kernels also arise in electromagnetic and integral-equation settings [26-29].

To address this gap, this paper develops a multidimensional Neutrix-based regularization framework and systematically studies its behavior under Lorentzian, Gaussian, polynomial, exponential, rational, and hyperbolic smoothing strategies. We derive dimension-dependent singular integral forms to analyze divergence explicitly in terms of spatial dimension and kernel order under radial symmetry. We then construct a systematic Neutrix subtraction procedure to extract finite parts from divergent regularized integrals. Analytical and numerical comparisons are performed across multiple regularization kernels to evaluate smoothing profiles, finite-part behavior, convergence properties, numerical integration error, and validation against benchmark cases. The novelty of the present work is rather not in the introduction of the Neutrix calculus itself, but in its systematic extension to multidimensional radial singular integrals along with a

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comparative numerical investigation of several regularization kernels. This work includes dimensional dependence and finite-part extraction in the same framework and compares kernels across dimensions, unlike studies that focus principally on quadrature or problem-specific regularization.

Materials And Methods

Theoretical foundations: multidimensional singular integrals and Neutrix analysis

Neutrix calculus has been broadly applied to normalize singular integrals and distribution products. In earlier contributions by Tiwari (2006, 2007, 2008) proved the Neutrix product framework was established for distributions and generalized functions. Current expansions [30-33] have advanced convolutional compositions and integral formulations on manifolds.

Understanding Singular Behavior in Higher Dimensions

Singular integrals in higher dimensions demonstrate more complex divergence configurations and stand up to their one-dimensional equivalents. The divergence predictably depends on both the longitudinal dimension d and the power n of the singular kernel.

Consider the generalized form of a singular integral over a domain $\Omega \subset \mathbb{R}^d$:

$$I = \int_{\Omega} \frac{1}{r^{2n}} dV$$

where

r is the radial distance from the origin (assumed to be the singular point).

One Dimension (1D): A representative 1D case is:

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{|x|^n} dx, \quad n > 0$$

This integral diverges at $x = 0$ for $n \geq 1$ and is only integrable when $n < 1$.

Two Dimensions (2D):

In two dimensions, we define the singular integral as:

$$I_2 = \int_{\mathbb{R}^2} \frac{1}{(x^2 + y^2)^n} dx dy$$

Using polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, the integral becomes:

$$I_2 = \int_0^{2\pi} d\theta \int_0^{\infty} \frac{r dr}{(r^2)^n}$$

Simplifying, we obtain:

$$I_2 = 2\pi \int_0^{\infty} r^{1-2n} dr$$

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This expression converges if

$$1 - 2n > -1 \Rightarrow n < 1$$

Three Dimensions (3D): The singular integral in three dimensions is given by:

$$I_3 = \int_{\mathbb{R}^3} \frac{1}{(x^2 + y^2 + z^2)^n} dx dy dz$$

Switching to spherical coordinates:

$$x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi$$

the integral transforms into:

$$I_3 = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\infty \frac{r^2 dr}{(r^2)^n}$$

Simplifying:

$$I_3 = 4\pi \int_0^\infty r^{2-2n} dr$$

This integral converges when

$$2 - 2n > -1 \Rightarrow n < \frac{3}{2}$$

Thus, the integral diverges for $n \geq \frac{3}{2}$ but converges for $n < \frac{3}{2}$.

As dimensions increase, the range of suitable n values for convergence becomes more obstructive. This means that singularities become new critical points as we move into higher-dimensional intervals.

Formalizing the Multidimensional Neutrix Limit

Using Neutrix calculus, we resolve these divergences by analytically excluding divergent parts and assigning them finite values. This provides a systematic method, and thus we can handle divergent integrals in a mathematically stable framework.

Theorem (Neutrix Regularization of Singular Integrals)

The regularized singular integral is defined as

$$I(\epsilon) = \int_{\mathbb{R}^d} \frac{dV}{(r^2 + \epsilon)^n}$$

convergence occurs in the classical sense when

$$d - 2n > 0 \Rightarrow n < \frac{d}{2}$$

This integral diverges logarithmically when the critical boundary condition is reached when $n = \frac{d}{2}$, so in this situation the Neutrix limit provides a generalized method that helps separate the finite contribution from the divergent part.

Asymptotically,

$$I(\epsilon) \sim C(\epsilon) + \log(R)$$

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where R represents the integration cutoff or outer boundary, and $C(\epsilon)$ captures behaviors near the origin.

Hadamard-Type Finite Part and Asymptotic Series

The Hadamard finite part provides an analytical foundation for the Neutrix abstraction, and these methods are particularly useful when dealing with integrals that diverge with power-law behavior.

A regularized integral can be expanded asymptotically if $\epsilon \rightarrow 0$:

$$\lim_{\epsilon \rightarrow 0} I_{\text{reg}}(\epsilon) = \sum_{k=1}^M C_k \epsilon^k + C_0 \log(\epsilon) + I_{\text{finite}} + O(\epsilon^p)$$

where:

C_k are coefficients representing the singular asymptotic terms,

$C_0 \log(\epsilon)$ corresponds to the logarithmic divergence,

I_{finite} is the desired finite contribution,

$O(\epsilon^p)$ includes negligible terms such as $\epsilon \rightarrow 0$.

This expansion makes it possible to isolate the finite contribution through Neutrix subtraction once the singular behavior has been identified asymptotically.

Neutrix Limit in Singular Integral Regularization

To invert all divergent parts, we formally defined the Neutrix limit as:

$$N - \lim_{\epsilon \rightarrow 0} I_{\text{reg}}(\epsilon) = I_{\text{finite}}$$

This means that the divergent terms are removed and only the finite contribution remains, and because we can clearly separate divergent and finite contributions, this approach preserves reliability in multi-dimensional problems.

Regularization and Damping Strategies for Controlling Singularities

Regularization methods provide a systematic approach to controlling divergent integrals. These methods modify the behavior of the integrand near singular points. The central idea is to introduce a smoothing parameter ϵ to moderate the singular behavior and make it possible to isolate the finite contributions through analytical or numerical procedures.

In this section, we examine multiple regularization approaches, which have distinct smoothing profiles and their suitability varies depending on the mathematical or physical context. Their common purpose is to support Neutrix-based subtraction, where the dominant divergent part is systematically separated from the finite remainder, and we also distinguish two broad classes of singularity-control strategies. The first class is algebraic regularizations, which directly preserve near-origin asymptotic character. These methods provide the most natural setting for finite-part extraction because they ensure controlled behavior while retaining the underlying mathematical structure of the singularity. The second class is damping-based kernels, which aggressively suppress singular behavior through exponential or hyperbolic attenuation. These kernels are useful for numerical smoothing and qualitative comparison because they rapidly reduce divergence. But these do not belong to the asymptotic class of the original singular kernel and sometimes over-suppress finite contributions.

Primary Algebraic Regularizations

The primary regularization methods considered in this study have the main objective of preserving the algebraic structure of the singular kernel. These methods provide a natural and mathematically consistent framework where Neutrix-based finite-part comparison becomes systematically possible.

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Lorentzian Regularization

Definition: Operating Lorentzian regularization converts the singular integral kernel, which attains its behavior near the singular point, more practicable and mathematically well defined.

$$f_{\text{Lor}}(r) = \frac{1}{(r^2 + \epsilon)^n}$$

This function smooths the singularity at $r = 0$ by leading the small parameter ϵ , counteracting divergence.

Lorentzian regularization approaches a power-law degeneration amended by an easy limit ϵ . This method is standard in quantum field theory [18], where it simulates communicators, behavior and also confirms convergence without preceding sharp cutoffs [21]. Its algebraic extension domains are long-range interfaces.

Rational Regularization

Definition:

$$f_{\text{Rat}}(r) = \frac{1}{(r^2 + \epsilon^2)^n}$$

The rational kernel is symmetric for a singular, function and initiates an algebraically decaying modification, maintaining the equality between sharp cutoff and long-range behavior, making it well-suited for wave propagation and potential theory [24].

Polynomial Regularization

Definition:

$$f_{\text{Poly}}(r) = \frac{1}{(r^2 + \epsilon r + \epsilon^2)^n}$$

This method initiates a polynomial smoothing that mixes together the behavior of Lorentzian and rational configurations. The term ϵr maintains additional tunability and consents to a more nuanced control of singular behavior in the integration domain.

Overall, these algebraic regularizations retain the original algebraic character of the kernel while regularizing the singularity. This makes them most suitable for direct comparison to Neutrix-subtracted finite parts, and their role is fundamental in contrast to damping-based kernels.

Alternative Damping-Based Kernels

In addition to algebraic regularizations, we also consider some damping-based kernels, which are primarily used for qualitative comparison. These kernels more strongly suppress the near-origin behavior of singularities and illustrate how different smoothing profiles influence the numerical integral. Their primary role is to rapidly reduce divergence and provide numerical smoothing. Exponential kernels aggressively damp singularities through fast attenuation, while hyperbolic kernels introduce a smooth cutoff that ensures stability in transition zones. These approaches are very useful for computational experiments and qualitative visualization, as they immediately control the impact of divergence.

However, an important limitation is that damping-based kernels do not preserve the local asymptotic structure of the original singular kernel. Therefore, they are not a reliable basis for finite-part consistency and Neutrix subtraction. They sometimes over-suppress finite contributions, distorting analytical clarity. In this study, we include damping-based kernels as complementary tools - their role is primarily to illustrate comparative numerical analysis and qualitative smoothing, not to provide a primary framework for finite-part validation.

Gaussian Regularization

Definition:

$$f_{\text{Gauss}}(r) = \frac{e^{-r^2/\epsilon}}{r^{2n}}$$

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The Gaussian function is very important for quickly becoming major near the origin, as a result of its exponential decay. Even if it is also useful for obtaining local features, the method may control relevant contributions, especially in problems regulated by comprehensive dynamics or stretched singularities [23].

Exponential Regularization

Definition:

$$f_{\text{Exp}}(r) = \frac{e^{-r/\epsilon}}{r^{2n}}$$

This system is similar in nature to the Gaussian method, except that the reduction is gradual rather than linear exponential decay. A rapid - but not immediate - conquest with a finite propagation speed is required, which is why this formula is most commonly used to model boundary-layer phenomena.

Hyperbolic Regularization

Definition:

$$f_{\text{Hyp}}(r) = \frac{\tanh(r/\epsilon)}{r^{2n}}$$

This method uses a hyperbolic tangent function to smoothly dominate singular behavior near the origin and also stabilizes the structure of the function as it extends away from the origin, providing a middle ground between sharp and smooth regularization. We have used this method mainly for real systems in boundary layers or conversion zones where controlled transitions are required.

How to cite this article:

Tiwari C, Shukla A, Rastogi S (June 17, 2026) Singular Integral Regularization and Neutrix Subtraction: Analytical and Numerical Approaches. Cureus J Eng 3 : es44388-026-00125-3. DOI <https://doi.org/10.7759/s44388-026-00125-3>

Method	Suppression Profile	Decay Type	Remarks
Lorentzian	Smooth, algebraic	Power-law	Primary algebraic regularization; preserves asymptotic character
Rational	Balanced, algebraic	Power-law	Primary algebraic regularization; balanced smoothing
Polynomial	Tunable, polynomial	Hybrid	Primary algebraic regularization; tunable near-origin behavior
Gaussian	Rapid, exponential	Exponential	Alternative damping kernel; strong local suppression
Exponential	Moderate, exponential	Exponential	Alternative damping kernel; finite-range attenuation
Hyperbolic	Smooth cutoff near 0	Sigmoid-like	Alternative damping kernel; smooth transitional damping

TABLE 1: Comparison of algebraic regularizations and alternative damping-based kernels used for singularity control

This collection of regularization methods provides experts with the flexibility to modify their methodology based on the specific physical or mathematical environment. As shown in Table 7, Lorentzian and Rational methods preserve long-range interactions, while Gaussian and Exponential methods introduce strong local suppression. In the next part, we will calculate singular integrals in different dimensions to which functions can be applied, and our focus is on how these methods isolate finite parts through Neutrix, subtraction and systematically remove divergent behavior.

Materials and methods

In this study, we used analytical derivations which are based on radial coordinate transformations, asymptotic finite-part extraction, and Neutrix subtraction. Numerical evaluations were implemented using Python and the *scipy.integrate.quad* adaptive quadrature routine. The computational experiments compared standard and regularized singular integrals in one, two, and three dimensions, estimated multiple smoothing kernels, and measured benchmark accuracy, convergence with respect to ϵ , cross-method consistency, and also numerical integration error. The analysis is limited to radially symmetric "singular" kernels of the form $r^{-\alpha}$ and within limited domains. The regularized integral is expanded in powers of ϵ and the finite part is extracted afterwards. In the Neutrix sense, the terms that diverge as $\epsilon \rightarrow 0$ (such as inverse powers and logarithmic terms) are omitted, and the constant term, the remainder, is considered the finite part.

Artificial Intelligence Declaration

Large Language Models were used only for language polishing and grammatical correction.

Numerical Evaluation and Initial Comparative Results

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We numerically calculate singular integrals in one, two, and three dimensions to identify the practical efficiency of different regularization schemes. For high-accuracy numerical integration, we calculated integrals using Python's `scipy.integrate.quad` function, which allows adaptive quadrature. Our main focus is to compare raw integral values (regularized and non-regularized), understand the impact of dimensionality, and estimate the exploitative impact of each regularization method.

One-Dimensional Singular Integral

Let us start with the one-dimensional integral:

$$I_1 = \int_a^b \frac{1}{x^n} dx$$

We integrate from $a = 1$ to $b = 10$ to avoid singularities because when $n = 1.2$, the integrand diverges at $x = 0$. A regularized version is also calculated with the lower bound replaced by a small parameter: $\epsilon = 0.01$.

Integral Type	Range	Value
Standard Integral	[1, 10]	1.845213
Regularized Integral	[0.01, 10]	9.404645

TABLE 2: Comparison of standard vs. regularized integrals

As shown in Table 2, the value of the integral increases significantly when approaching the singular region $x = 0$, clearly highlighting the importance of careful regularization or subtraction methods.

Two-Dimensional Radial Integral

We analyze a rotationally symmetric integral in polar coordinates:

$$I_2 = 2\pi \int_{r_{\min}}^{r_{\max}} r^{1-2n} dr$$

For $n = 1.2$, the singularity occurs at $r = 0$. We consider two evaluations shown in Table 3.

Integral Type	Range	Value
Fixed Limits	[1, 10]	9.454510
Regularized Integral	[0.01, 10]	92.857095

TABLE 3: Comparison of standard vs. regularized integrals

As shown in Figure 1, the integrand diverges near the origin as the regularized form reveals a dramatic increase in magnitude, and this behavior clearly highlights the strong impact of radial singularities and also how the parameter ϵ plays a crucial role in mitigating the divergence.

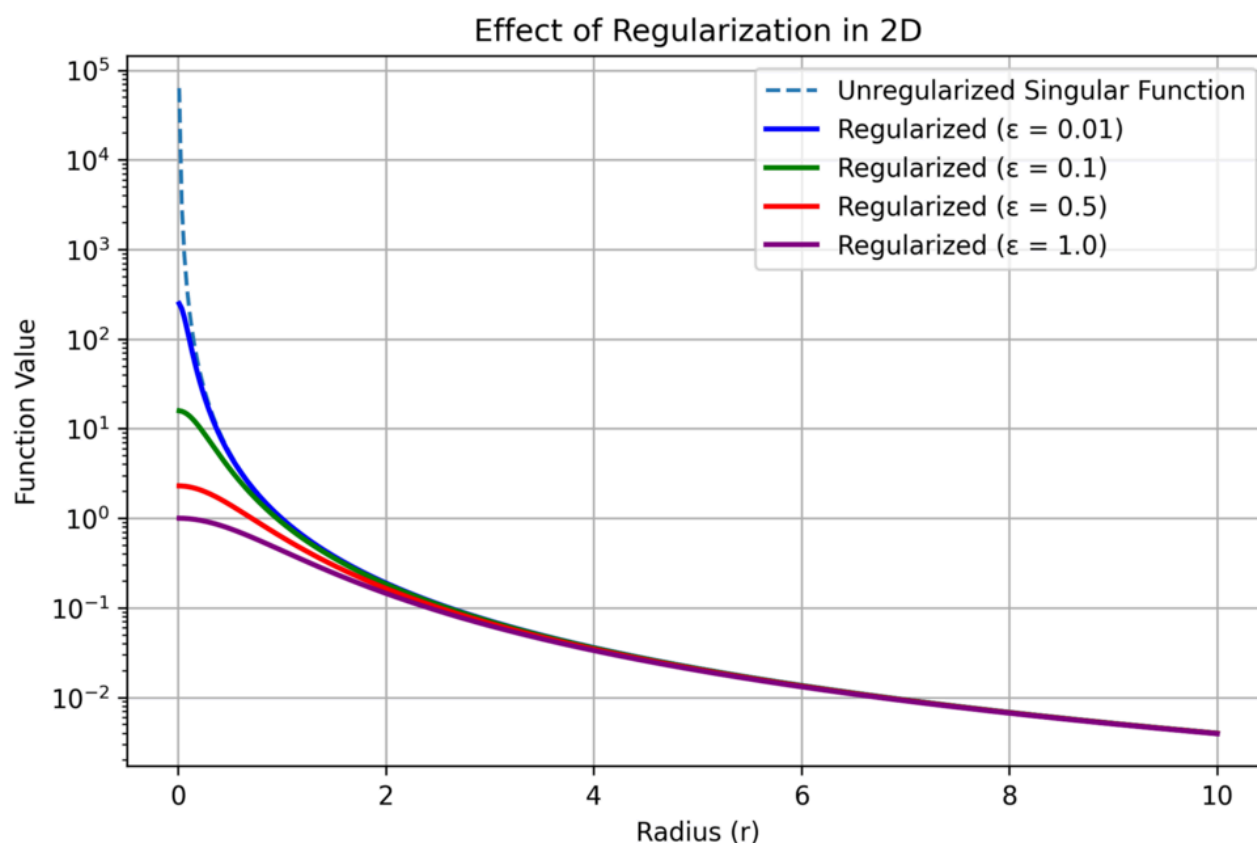


FIGURE 1: Log-scale plots compare the unregularized singular function with its regularized versions for multiple ϵ values, and these plots clearly illustrate the divergence near $r = 0$, and how regularization smooths out the singular behavior

Three-Dimensional Radial Integral

Under spherical coordinates, the singular integral assumes the form

$$I_3 = 4\pi \int_{r_{\min}}^{r_{\max}} r^{2-2n} dr$$

The singularity occurs at $r = 0$ when $n = 1.2$. In this case, we consider two evaluations shown in Table 4.

Integral Type	Range	Value
Fixed Limits	[1, 10]	62.435420
Regularized Integral	[0.01, 10]	82.057897

TABLE 4: Analytical contrast of standard and regularized 3D integrals

As shown in Figure 2, the log-scale plot clearly illustrates how regularization smooths out singular behavior near $r = 0$ and controls divergence in the 3D domain.

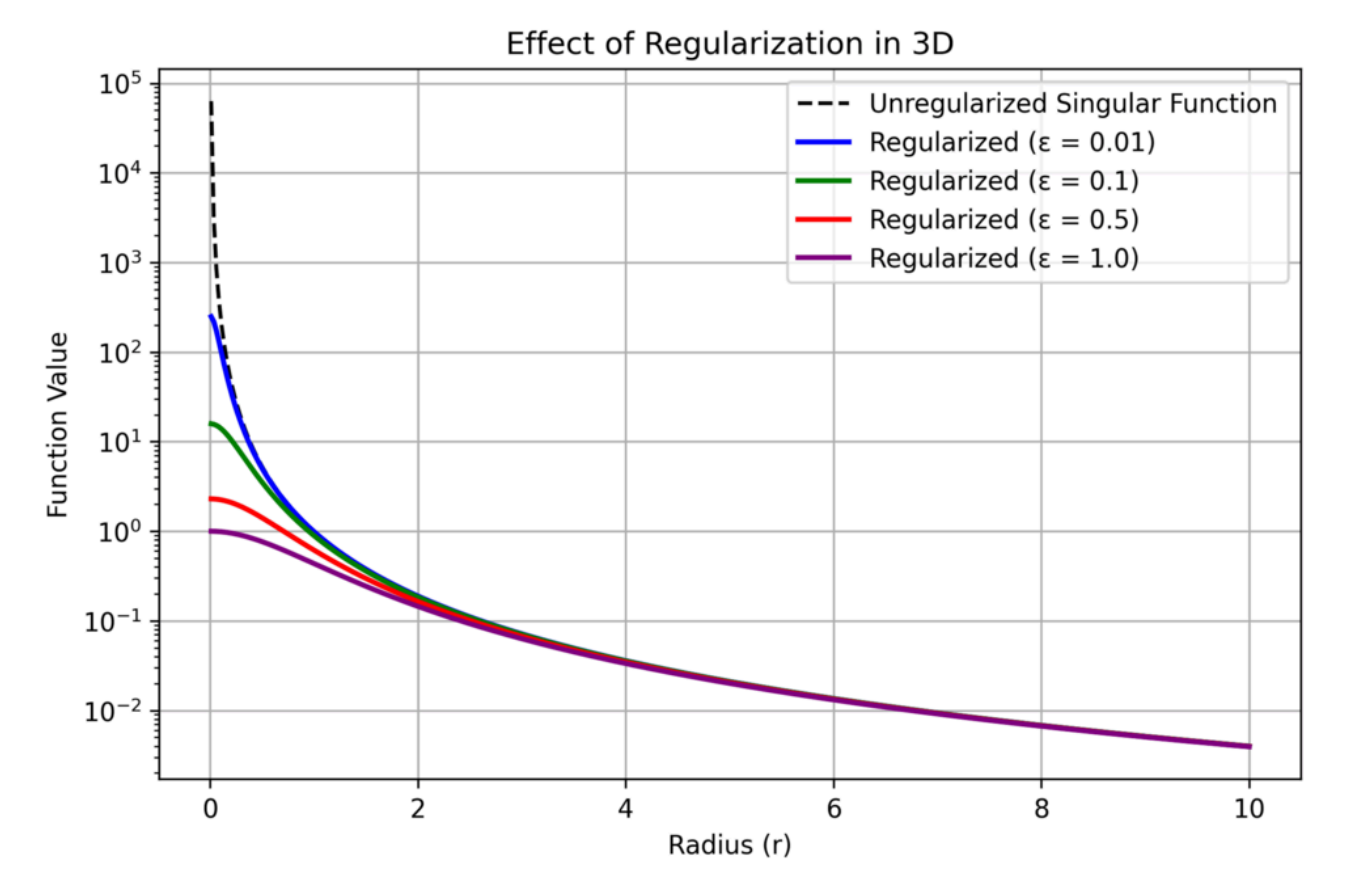


FIGURE 2: Log-scale plot of the regularized singular function, visualizing singular behavior in 3D and the smoothing effect of regularization near $r = 0$

Comparative Analysis of Regularization Approaches

We have made a clear distinction between algebraic regularizations and damping-based kernels for validation purposes. Algebraic regularizations have been used as the primary reference class for finite-part consistency, as they directly preserve the asymptotic structure of the singular kernel and provide a natural analytical basis for Neutrix subtraction. However, they mainly illustrate how stronger attenuation impacts numerical integral values and finite-part extraction.

Damping-based kernels aggressively suppress divergence and are useful for qualitative comparison, but have the drawback that they do not belong to the original asymptotic class and sometimes over-dampen finite contributions. Damping-based kernels are included as complementary comparative tools to clearly highlight the effects of numerical smoothing and qualitative visualization, while algebraic methods are treated as benchmark references.

Now we compare the raw integral values produced by different regularization functions, when $n = 1.2$ and $\epsilon = 0.01$ are taken, within the same 3D domain.

Regularization Type	Raw Integral Value
Lorentzian	76.791266
Gaussian	3.403055
Polynomial	80.304942
Exponential	4.887252
Rational	78.411604
Hyperbolic	80.304942

TABLE 5: Comparison of regularization methods and their integral values

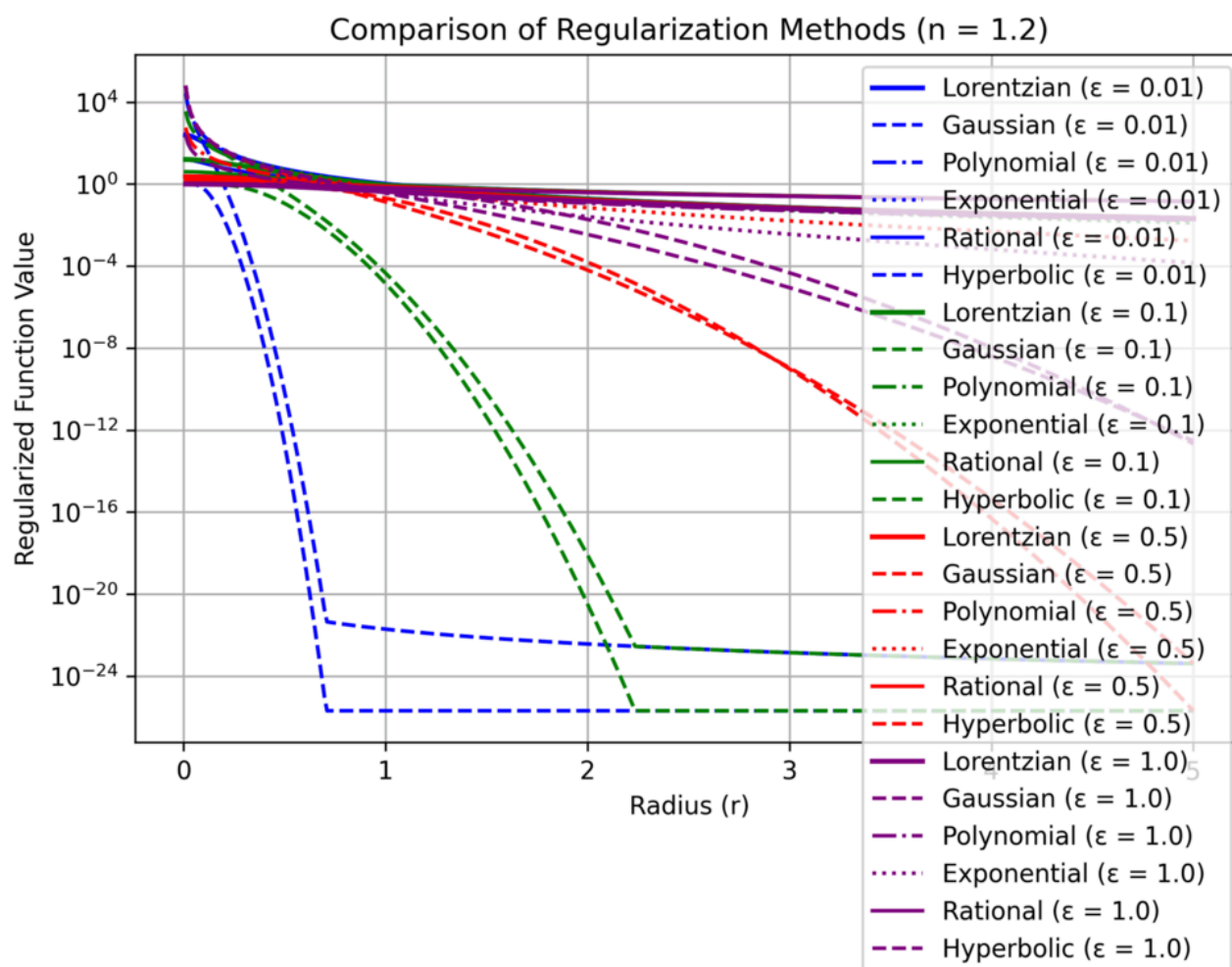


FIGURE 3: Comparison of regularization schemes on a log-scale across r

As shown in the Table 5 and Figure 3, sometimes Gaussian and Exponential regularizations significantly dampen the integrand near $r = 0$, which can distort the finite part. On the other hand, Polynomial, Rational, and Lorentzian forms retain more of the structure of the original function. They also offer the ability to subtract divergences while preserving key features.

Neutrix Subtraction: Finite Part Extraction

The physically important part of every divergent integral is to completely isolate the finite part. We apply Neutrix subtraction [19,24], and in this process, the major singular contributions are identified and analytically subtracted from the standardized result. To do this, we use the following approximations for the divergent parts:

Power-law divergence (e.g., Lorentzian, Rational)

The singular integral follows a power-law decay:

$$I_{\text{singular}} \approx \frac{4\pi}{3-2n} \cdot \epsilon^{3-2n}$$

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This approach is useful in preserving the structure of the original function because the decay depends on two parameters - the regularization parameter ϵ and the exponent n .

Exponential-type divergence

For exponential suppression:

$$I_{\text{singular}} \approx \frac{4\pi}{\epsilon} \cdot e^{-\epsilon/R}$$

This form leads to rapid suppression to effectively mitigate singularity effects, but the drawback is that it sometimes over-dampens finite integral contributions.

Regularization Method	Raw Integral	Singular Part	Finite Part
Lorentzian	76.791266	1.321474	75.469792
Gaussian	3.403055	1.321474	-1251.977997
Polynomial	80.304942	1.321474	78.983468
Exponential	4.887252	1.321474	3.565778
Rational	78.411604	1.321474	77.090130
Hyperbolic	80.304942	1.321474	78.983468

TABLE 6: Comparison of regularization methods based on integral values

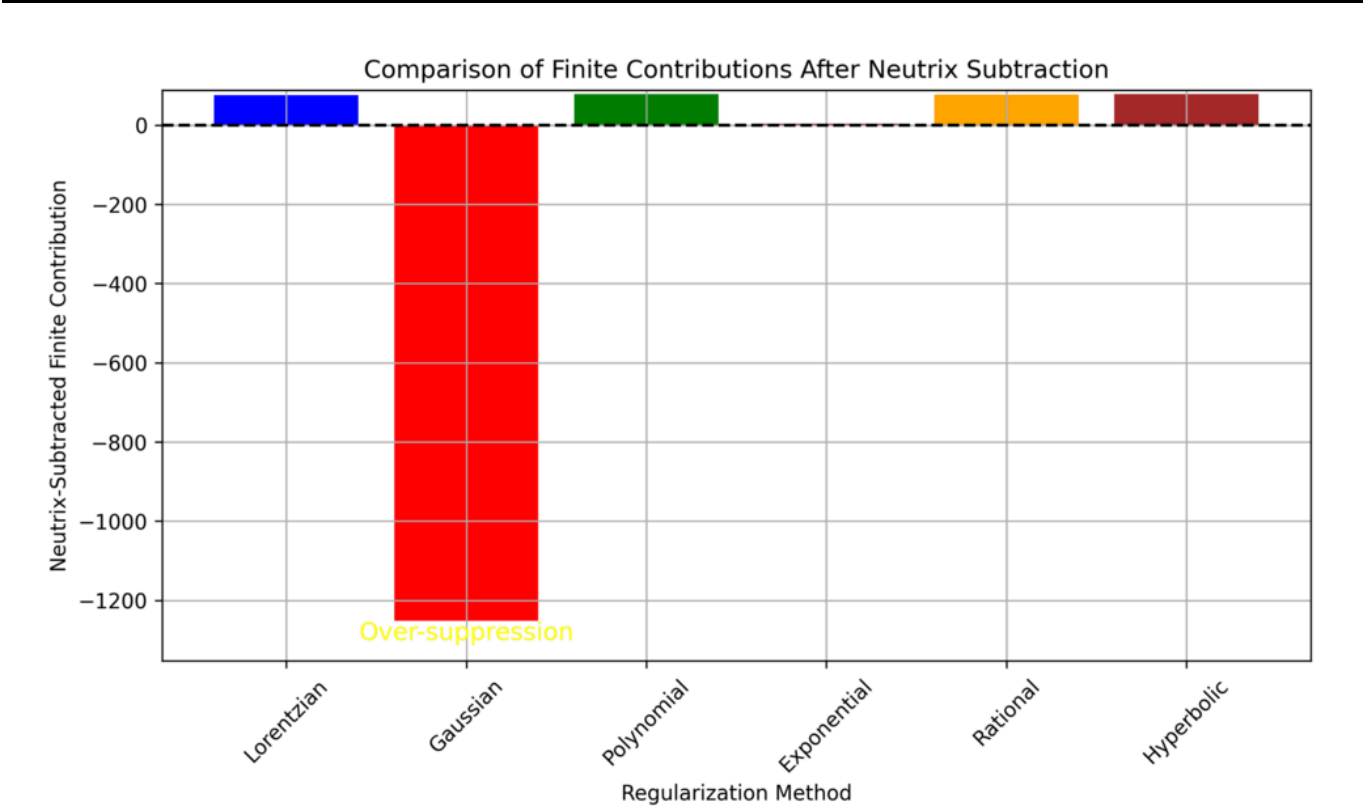


FIGURE 4: Bar plot comparing Neutrix-subtracted values across different regularization methods. This visualization clearly shows that Gaussian results appear unstable, which indicates over-suppression is occurring

As clearly shown in Table 6, different regularization methods produce different finite parts after Neutrix subtraction, and the Gaussian result appears unstable. This strongly negative finite-part value is obtained for Gaussian regularization because of the difference in algebraic form of the singular kernel and exponential damping. Gaussian damping is more effective at reducing the near-origin contribution than are algebraic damping kernels. This can leave an imbalance in the constant term following the subtraction. So, the asymptotic correction terms in the Gaussian and exponential regularizations can be different for each kernel. This observation becomes even more evident in the bar plot of Figure 4, where the Gaussian method shows over-suppression while the other methods give stable finite values.

Convergence Behavior Across Varying Exponents

Now we evaluate how the singular kernel exponent varies in the range $n \in [0.5, 3.5]$, with a step size of 0.2 to Neutrix-subtracted finite parts within the same 3D domain.

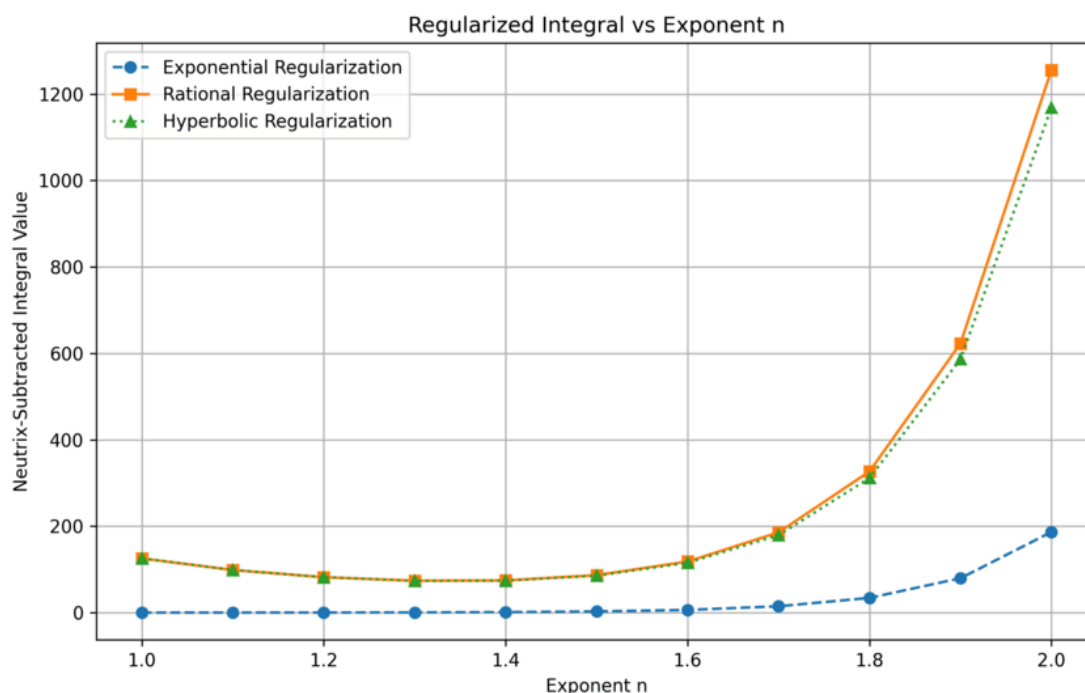


FIGURE 5: Line plot comparing neutrix-subtracted values for exponential, rational, and hyperbolic methods across different exponent values. This comparison clearly shows that the rational and hyperbolic forms show strong consistency, while the exponential method tends to suppress slightly more

Figure 5 shows that when $n < 1.5$, the finite-part estimates show stable behavior for the primary regularization methods. However, for $n > 2$, the Exponential and Gaussian regularizations show significant deviations, indicating over-suppression. In contrast, the Polynomial, Rational, and Hyperbolic methods behave well in every tested exponent range and consistently preserve the structure of the original function.

Numerical Error Analysis

We evaluate the numerical integration error (compiled from `scipy.integrate.quad`) for each regularization method as a function of the exponent n . This analysis reveals which methods consistently produce low errors and which methods become unstable as the exponent n increases. The main objective of this evaluation is to ensure computational robustness so that the results are reliable and reproducible.

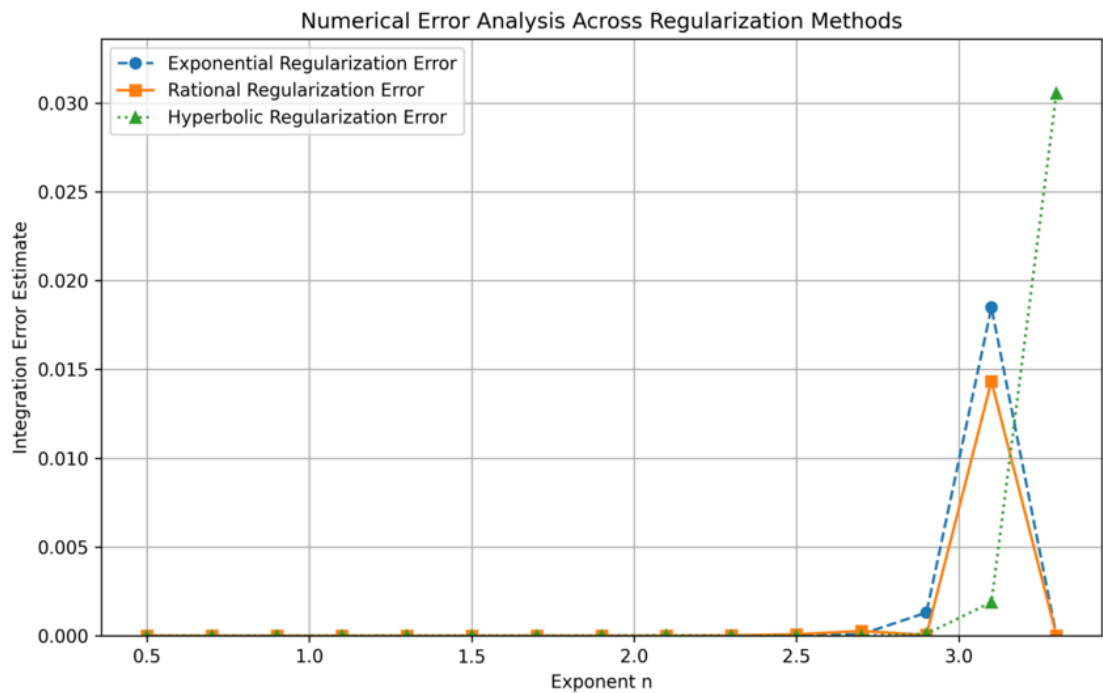


FIGURE 6: Numerical integration error across selected regularization methods, highlighting the behavior of exponential, rational, and hyperbolic kernels over the tested exponent range

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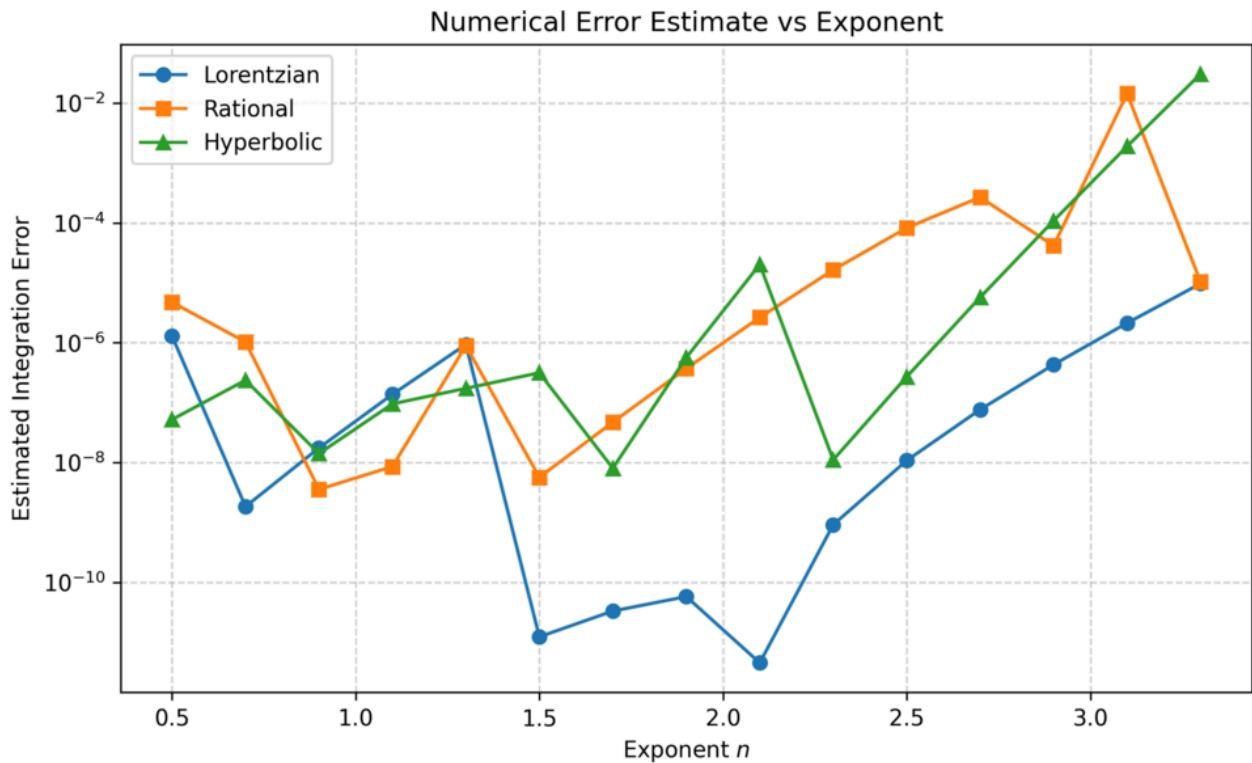


FIGURE 7: Estimated numerical integration error as a function of the exponent n for Lorentzian, rational, polynomial, and hyperbolic regularizations. Lorentzian errors remain smallest over most of the tested range, while the other methods show increased sensitivity for larger exponents

Figure 6 shows how the error behavior for a mixed set of regularization methods differs, with the exponential kernel having stronger sensitivity near larger exponents. Figure 7 focuses on the primary comparison class used in this paper such as Lorentzian, Rational, Polynomial, and Hyperbolic regularizations. Within this group, the Lorentzian method exhibits the smallest estimated numerical errors across most of the tested range, while the remaining methods remain numerically stable at moderate exponent values and become more sensitive as the singularity strength increases.

Results And Discussion

We applied multiple validation strategies to test the reliability of the proposed framework. Our main objective was to ensure that the extracted finite part is not just a numerical artifact resulting a particular kernel or discretizations, but a genuine mathematical quantity associated with the underlying singular integral.

Validation against closed-form benchmarks

First, we compared numerical results with analytically tractable benchmark cases, or regularized radial integrals for certain specific exponents and dimensions that give closed-form or asymptotic expressions. If $I_{\text{num}}(\epsilon)$ is the numerical value and $I_{\text{exact}}(\epsilon)$ is the analytical benchmark, then we defined the relative error as:

$$E_{\text{rel}}(\epsilon) = \frac{|I_{\text{num}}(\epsilon) - I_{\text{exact}}(\epsilon)|}{|I_{\text{exact}}(\epsilon)|}$$

Validation can only be successful if $E_{\text{rel}}(\epsilon)$ remains consistently small across ϵ and n .

ϵ	Numerical	Exact	Absolute Error	Relative Error	Quad Error
10−2	123.701935	123.701935	0.000000×10^0	0.000000×10^0	4.991314×10^{-8}
10−3	125.040750	125.040750	3.439027×10^{-12}	2.750325×10^{-14}	2.323097×10^{-7}
10−4	125.466440	125.466440	7.105427×10^{-14}	5.663210×10^{-16}	3.527877×10^{-7}
10−5	125.601298	125.601298	1.421085×10^{-14}	1.131426×10^{-16}	3.020899×10^{-10}

TABLE 7: Comparison between numerical quadrature and the closed-form benchmark for the Lorentzian regularization in the case $n = 1$

Table 7 clearly shows that there is essentially exact agreement between numerical quadrature and the closed-form benchmark, with relative errors remaining close to machine precision, which strongly supports the correctness of the computational implementation and confirms the reliability of the numerical integration procedure.

Convergence with respect to the regularization parameter

We computed the regularized integral for a decreasing sequence of regularization parameters $\epsilon = 10^{-1}, 10^{-2}, \dots, 10^{-6}$ to find the numerical stability of the Neutrix-subtracted quantity. For each value of ϵ , we evaluated the raw integral, the leading singular term, and the extracted finite part and we also calculated the consecutive differences

$$\Delta_k = |I_{\text{finite}}(\epsilon_k) - I_{\text{finite}}(\epsilon_{k+1})|$$

which provide a direct measure of convergence as $\epsilon \rightarrow 0$.

ϵ	Raw Integral	Singular Term	Finite Part	Quad Error
10−1	70.208809	5.260883	64.947926	3.827271×10^{-12}
10−2	76.791266	1.321474	75.469792	3.627653×10^{-7}
10−3	80.076283	0.331939	79.744344	1.467671×10^{-8}
10−4	81.723742	0.083379	81.640362	4.934922×10^{-12}
10−5	82.549574	0.020944	82.528630	4.926372×10^{-11}
10−6	82.963485	0.005261	82.958224	2.254263×10^{-9}

TABLE 8: Convergence of the Neutrix-subtracted finite part as the regularization parameter ϵ decreases

Table 8 clearly shows that as ϵ decreases, the Neutrix-subtracted finite part approaches a stable regime. The finite part is not perfectly stationary in the tested range, but the results show a clear convergence trend. This confirms that the subtraction captures the dominant divergent contribution and leaves a bounded finite remainder.

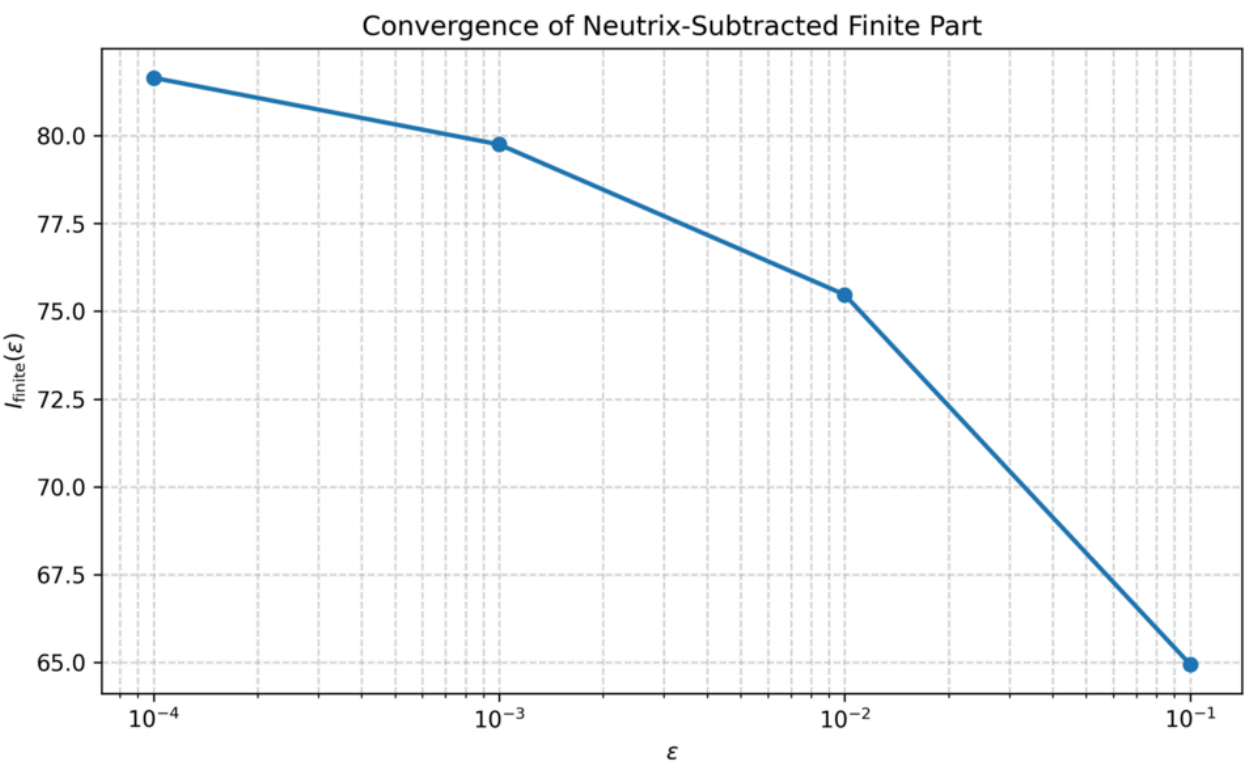


FIGURE 8: Convergence of the Neutrix-subtracted finite part as the regularization parameter ϵ decreases. The finite part approaches a stable regime as $\epsilon \rightarrow 0$, indicating a clear convergence trend

Figure 8 visually confirms the trend reported in Tables 8 and Tables 9, as the extracted finite part approaches a stable range, which strongly supports the numerical consistency of the Neutrix subtraction procedure as the regularization parameter decreases.

ϵ_k	ϵ_{k+1}	$ \bar{I}_{\text{finite}}(\epsilon_k) - \bar{I}_{\text{finite}}(\epsilon_{k+1}) $
10 ⁻¹	10 ⁻²	10.521866
10 ⁻²	10 ⁻³	4.274552
10 ⁻³	10 ⁻⁴	1.896018
10 ⁻⁴	10 ⁻⁵	0.888267
10 ⁻⁵	10 ⁻⁶	0.429595

TABLE 9: Decay of consecutive differences in the extracted finite part, showing numerical convergence of the subtraction procedure

The decreasing values in Table 9 provide direct numerical evidence that the extracted finite part becomes progressively less sensitive with respect to the regularization parameter. This monotone decay clearly supports convergence for the Neutrix-subtracted quantity. We believe a small residual drift still remains, which suggests that higher-order asymptotic corrections may provide more stabilization.

Cross-method consistency of the finite part

Different regularization kernels modify local singular behavior in different ways. But if the finite part is physically meaningful, it should remain approximately invariant across methods that preserve the asymptotic structure of the kernel.

Set of methods:

$$\mathcal{R} = \{\text{Lorentzian, Polynomial, Rational, Hyperbolic}\}$$

For sample mean and standard deviation computed:

$$\bar{I}_{\text{finite}} = \frac{1}{|\mathcal{R}|} \sum_{j \in \mathcal{R}} I_{\text{finite}}^{(j)}, \quad \sigma_{\mathcal{R}} = \left(\frac{1}{|\mathcal{R}|} \sum_{j \in \mathcal{R}} \left(I_{\text{finite}}^{(j)} - \bar{I}_{\text{finite}} \right)^2 \right)^{1/2}$$

If $\sigma_{\mathcal{R}}$ is small, then the finite part is cross-method consistent.

Method	Finite Part	Relative Deviation (%)
Lorentzian	75.469792	4.199549
Rational	80.245864	1.863140
Polynomial	78.747690	0.038624
Hyperbolic	80.649123	2.375032

TABLE 10: Comparison of finite parts obtained from algebraic regularization methods, together with relative deviation from the group mean

Table 10 clearly shows that algebraic regularizations produce mutually comparable finite-part values, where relative deviations remain consistently small with respect to the group mean and the polynomial method shows the closest agreement, which observation strongly supports cross-method consistency especially within the algebraic regularization class.

Method	Raw Integral	Quad Error
Gaussian	3.403055	9.596856×10^{-11}
Exponential	0.232916	1.465614×10^{-10}

TABLE 11: Qualitative comparison of strongly damping kernels. These methods are reported separately because they modify the near-origin asymptotic structure more aggressively than the primary algebraic regularizations

Table 11 clearly shows that Gaussian and Exponential kernels produce much smaller raw integral values by strongly suppressing singular contributions, which makes them useful for qualitative smoothing comparisons, but should not be used as the primary basis for finite-part consistency validation.

Discussion

The mathematical calculations and logical formulations that we present in this paper clearly highlight that the regularization approach plays a significant role in obtaining finite and physically important results, especially when singular integrals are involved in higher dimensions [1,9].

Consistency Across Regularization Methods

After Neutrix subtraction, regularization functions such as Lorentzian, Rational, Polynomial, and Hyperbolic mutually return comparable finite values impacts [19,24]. These methods preserve the asymptotic structure of the original singularity but smoothly modulate divergent behavior, and these stable results in the 1D, 2D, and 3D domains support their strength and reliability, especially for singular integral calculations in physical frameworks such as potential theory

and field propagation [26]. In contrast, Gaussian and Exponential regularizations are effective in controlling singularities near the origin but tend to over-suppress the integrand [23] so finite parts diverge considerably from other methods. These formulations are popular in fields such as signal processing and statistical physics because their smooth decay is useful, and although they render divergent integrals finite, practical difficulties arise when careful recalibration or modified subtraction techniques are required.

Dimensionality and Divergence Structure

Our study confirms that the intensity of deviations increases proportionally with increasing three-dimensional dimensionality, and this is clearly reflected in the shift in the starting point of deviations, which begin earlier in higher dimensions. Simply put, singularities appear more aggressively with increasing dimensionality, and their impact becomes more pronounced in both computational and physical modeling.

Convergence Conditions

The convergence behavior across various dimensions is as follows.

In 2D: The integral converges if:

$$n < 1$$

In 3D: The integral converges under:

$$n < \frac{3}{2}$$

These constraints ensure that singularities are adequately covered within bonded domains and finite results can be reliably extracted, and this trend clearly points out the need for more accurate regularization and subtraction methods in higher dimensions. This paper suggested that the Neutrix calculus, with its dimensionally adaptive structure, can be used as a flexible tool that can remove divergent terms across different dimensions.

$$I_{\text{reg}}(\epsilon) = \sum_{k=1}^M \frac{C_k}{\epsilon^k} + C_0 \log \epsilon + I_{\text{finite}} + O(\epsilon^p)$$

Its general form demonstrates how Neutrix subtraction systematically isolates finite contributions and preserves stability, eliminating divergence across multiple dimensions.

Convergence and Subtraction Accuracy

The Convergence plots for Rational and Hyperbolic methods clearly show that the Neutrix-subtracted values across the tested exponent range remain comparatively stable [19,24]. Without introducing sharp truncations, these methods preserve key features of the kernel, making the representation of fundamental physics more accurate. This stability proves that Neutrix subtraction not only systematically removes divergent terms but also preserves the physical interpretation - and thus establishes a reliable framework for higher-dimensional singular integral problems. The subtraction terms are methodically derived and their accuracy is reliable. This confirms the effective elimination of divergences in the validation section of the benchmark comparison. This establishes that the Neutrix calculus is not just a theoretical tool, but a numerically controllable method that is compatible with standard computational practices [19,24]. This approach forms a foundational framework for controlling singular integrals that is crucial in obtaining physically important finite results, especially for higher-dimensional problems [18,22].

Practical Implications

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These works provide a methodological framework to standardize singular integrals in different domains. Renormalization techniques [21,28] are needed when, in quantum field theory, Green's functions and propagators diverge over short distances. In electrostatics and potential theory, control distributions or field strengths show point singularities that involve boundary conditions or image control methods [26]. Also in materials science and fracture mechanics, when fields near crack tips repeatedly generate strong singularities that are modeled by asymptotic growths [22,26]. In all these cases, if a suitable regularization scheme is carefully selected and Neutrix-based subtraction is applied, finite measures are obtained that remain invariant under coordinate scaling and domain truncation-essential for confirming physical consistency. Neutrix calculus [19,24] provides a stable and scalable framework that is effective in isolating finite parts from multidimensional singular integrals. Rational, Polynomial, and Hyperbolic regularizations provide a balanced approach that maintains smooth dominance and analytical controllability. Gaussian and Exponential methods, despite their efficiency, sometimes require modified subtraction approaches to confirm consistency. The numerical accuracy is highest when the chosen regularization form closely aligns with the asymptotic structure of the singular kernel. This makes these approaches robust in practical computational settings and strengthens reliability. The drawback of the present study is the absence of validation based on application-specific models obtained from boundary element methods, fracture mechanics, electromagnetics, or quantum field theory, but on the controlled radial benchmark integrals. For this reason, the results should be considered as a validation of the numerical stability of the proposed framework for radial test cases, but further validation for broader physical applications is necessary.

Conclusions

A Neutrix framework on the regularization of the higher dimensional radial singular integrals and extraction of finite parts from divergent expressions is presented in this study. The numerical solutions demonstrate that the singular region becomes more prominent when the lower limit of integration approaches the origin. For example, when $n = 1.2$, the one-dimensional integral increases from 1.845213 over the closed interval $1 \leq r \leq 10$ to 9.404645 over $0.01 \leq r \leq 10$. Likewise, the radial integral changes from 9.454510 to 92.857095 for the 2D version and 62.435420 to 82.057897 for the 3D version. The results presented here agree that the strength of the singularity as well as the spatial dimension have a significant influence on the value of the integral.

In the proposed Neutrix framework several regularization strategies were compared. The raw values for the integral were obtained as 76.791266, 78.411604, and 80.304942 for Lorentzian, Rational, and Polynomial/Hyperbolic kernel, respectively, in the three-dimensional case. Their respective finite parts were also relatively stable, with 75.469792, 77.090130, and 78.983468. The raw integral values for the Gaussian and Exponential damping kernels were much smaller, 3.403055 and 4.887252, respectively. This shows the importance of these kernels in suppressing the contribution from the near-origin, and the need for kernel specific correction terms for consistent extraction of the finite parts.

For both numerical and closed-form results, the computed values are very close to the benchmark values, even for as small an ϵ as 10^{-5} . The same can be seen in the differences between the finite parts extracted, when ϵ is decreased, in the convergence analysis. The observations confirm the numerical reliability of the proposed implementation of the considered cases of radial benchmark in this study.

Overall, the outcomes suggest that algebraic-type regularizations are comparatively stable and generally yield relatively accurate finite-part estimates of the benchmark radial singular integrals examined here, especially for the Lorentzian, Rational, Polynomial and Hyperbolic kernels. Gaussian and Exponential kernels are good smoothing functions, but can over-smooth singular contributions without adding correction terms. Hence, general statements about the engineering or physical implementation should be taken with a pinch of salt until further testing on application-specific benchmark problems is conducted.

Future work

This work opens up several promising directions for future research, such as the extension of Neutrix-based methods to nonlinear integral equations, including boundary integral formulations in fluid dynamics and elasticity problems. Adaptive regularization objectives can also enable the development of parameterized or data-driven structures that

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dynamically adjust to local singularity reports. Furthermore, symbolic-neural hybrids, which integrate symbolic computation with machine learning, can enhance the detection and subtraction of divergent terms in large-scale simulations - improving both efficiency and reproducibility. Another opportunity is to combine Neutrix subtraction with multiscale and hybrid models, where singular and regular methods can be extended simultaneously. In the future, the current structure will be expanded to well-known problems in the field of boundary element methods, fracture mechanics, electromagnetics, and quantum field theory. For Gaussian regularization and Exponential regularization, there is also more work to be done to develop subtraction formulas for specific kernel.

Final remark

This research establishes a balanced framework for merging classical mathematical methods (such as analytical integration and theoretical constructs) with modern computational techniques (numerical simulation and algorithmic tools) and its main aim is to systematically handle singular high-dimensional integrals. Singular integrals are problematic because the results are divergent, and hence the role of Neutrix calculus is crucial here - it isolates finite and physically meaningful contributions by analytically subtracting the divergent parts. This approach has a double advantage: first, theoretical clarity: mathematical proofs and derivations become consistent and logically sound, and second, practical computation: numerical algorithms produce reliable and stable results, which can also be directly used in applied physics and engineering models. Therefore, Neutrix calculus is a valuable toolkit addition for applied mathematics and computational physics as it ensures both aspects - rigorous theory and reliable numerical implementation.

Additional Information

Author Contributions

All authors have reviewed the final version to be published and agreed to be accountable for all aspects of the work.

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Disclosures

Human subjects: All authors have confirmed that this study did not involve human participants or tissue. **Animal subjects:** All authors have confirmed that this study did not involve animal subjects or tissue. **Conflicts of interest:** In compliance with the ICMJE uniform disclosure form, all authors declare the following: **Payment/services info:** All authors have declared that no financial support was received from any organization for the submitted work. **Financial relationships:** All authors have declared that they have no financial relationships at present or within the previous three years with any organizations that might have an interest in the submitted work. **Other relationships:** All authors have declared that there are no other relationships or activities that could appear to have influenced the submitted work.

Data Availability Statements

The datasets (and/or code) supporting this study are available from the corresponding author upon reasonable request.

How to cite this article:

Tiwari C, Shukla A, Rastogi S (June 17, 2026) Singular Integral Regularization and Neutrix Subtraction: Analytical and Numerical Approaches. *Cureus J Eng* 3 : es44388-026-00125-3. DOI <https://doi.org/10.7759/s44388-026-00125-3>

Acknowledgements

We would like to express our sincere gratitude to the organizers of ICRAME 2026: International Conference on Recent Advances in Multidisciplinary Engineering for giving us the opportunity to present our research on April 1–2, 2026. We are also thankful to Maharishi University of Information Technology for providing a platform to share ideas, interact with fellow scholars, and contribute to meaningful academic discussions by hosting such a prestigious event.

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