Cureus Journal of Computer Science

Open Access Original Article

Duality Between Lines and Points

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Received 03/12/2025
Review began 03/18/2025
Review ended 08/12/2025
Published 08/21/2025

DOI:
https://doi.org/10.7759/s44389-025-03728-9

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Abstract

In Computational Geometry, there are several notions of duality between lines and points. In this note, it is shown that *all these can be studied in a unified way*. There is a common generalisation of all these notions. Most interesting properties of various notions of duality are also valid for this generalisation.

However, there are two properties, which cannot simultaneously be true. It is also shown that either dual mapping can be its own inverse or it can preserve relative order (but not both).

Generalisation of notion of duality to higher dimensions is also discussed. An elementary and very intuitive treatment of relationship between arrangements (in Computational Geometry) in *d*+1 dimensions and searching for *k*-nearest neighbour in *d*-dimensions is also given.

Categories: Algorithm Analysis, Algorithm Design, Data Structures
Keywords: duality, arrangements, nearest neighbours, computational geometry, point-line maps

Introduction

A non-vertical line is determined by two parameters, for example, m and c in y = mx + c and a and b in $\frac{x}{a}+\frac{y}{b}=1$. Here, m and c in the first equation and a and b in the second equation are arbitrary real numbers or parameters (these numbers or parameters, as far as this note is concerned, have no other significance), and a point in 2D also requires two parameters (coordinates, e.g., x and y in Cartesian and r and θ in polar).

There are several notions of duality between lines and points in Computational Geometry, in computer science. These notions make the proofs and algorithms easier to visualise; we will see an example in the last section of Materials and Methods. Various authors suggest different ways to map points to lines and vice-versa:

Lee and Ching [1] and JaJa [2] show that point (r,s) is mapped to the line y = rx + s, and the line y = mx + c is mapped to the point (-m,c).

O'Rourke [3] suggests two mappings: the first maps line y = mx + c to point (m,c) and vice versa, and the other, which he actually uses, maps the point (r,s) to the line y = 2rx-c and vice versa.

Berg et al. [4] show that point (r,s) is mapped to the line y = rx-s, and dual of the line y = mx+c is the point (m,-c).

Chazelle et al. [5] show that point (r,s) is mapped to the line rx + sy + 1 = 0 and vice versa.

In this note, it is shown that all these can be studied in a unified way. Let us look at general dual mapping in which point (r,s) is mapped to line $y=\alpha rx+\beta s$, for constants α and β, and a line y = mx + c gets mapped to point $(\mu m, \lambda c)$, for constants μ and λ.

Most interesting properties are independent of specific choices of these constants. We also generalise the notion of duality to higher dimensions. Finally, an elementary and very intuitive treatment of relationship between arrangements in *d*+1 dimensions and searching for k-nearest neighbour in d-dimensions is described.

Materials And Methods

Dual mapping

In the rest of this section, it is shown that either dual mapping can be its own inverse or it can preserve relative order (but not both).

How to cite this article
Saxena S (August 21, 2025) Duality Between Lines and Points . Cureus J Comput Sci 2 : es44389-025-03728-9. DOI https://doi.org/10.7759/s44389-025-03728-9

O'Rourke [3] observes the following:

Observation 1: There is a one-to-one correspondence between all non-vertical lines and all points in the plane.

Remark: This property will not be true for more general "dual" lines like $y = (ap + bq)x + (cp + dq)$.

We will like the following property to be true for the mapping:

If a point p lies on a line L, then the dual of line L, say d(L) (a point), should lie on the dual of point p, say d(p) (which is a line d(p)).

A general line through point (r,s) will be y − s = m(x − r). This line gets mapped to a point $(\mu m, \lambda(s - mr))$. As this point must lie on line $y = \alpha rx + \beta s$, we get $\lambda(s - mr) = \alpha(\mu m) + \beta s$ or $s(\lambda - \beta) = m(\alpha\mu + r\lambda)$ or $\lambda = \beta$ or $\alpha\mu = -\lambda$, or $\alpha\mu = -\beta$.

Thus, the permissible transforms will map point (r,s) to the line $y = \alpha(rx - \mu s)$ and will map line y = mx + c to the point $(\mu m, -\alpha\mu c)$.

Lemma 1: If d( ) is a permissible dual transform, then a point p lies on a line L, then the dual of line L, d(L), lies on d(p), the dual of point p.

As in O'Rourke [3], we have:

Corollary: Two lines L_1 and L_2 intersect in a point P, if dual d(P) passes though $d(L_1)$ and $d(L_2)$.

As we have two free parameters, we can impose additional "desirable" conditions. Two of the popular desirable conditions are as follows:

1. If the dual of a point p is the line L, then the dual of the line L should be a point p.

2. If the point p lies above the line L, then the dual of p should lie above the dual of L.

Let us first look at the first condition. The dual of line $y = (\alpha r)x + (-s\mu\alpha)$ will be the point $(\mu(\alpha r), (-\mu\alpha)(-s\mu\alpha))$. For this point to be (r,s), we need $\mu\alpha = 1$.

Lemma 2: If $\alpha\mu = 1$, then dual d is its own inverse.

Remark: These transforms have only one free parameter. Point (r,s) gets mapped to line $(y = \alpha rx - s)$ and line y = mx + c gets mapped to the point $(\frac{1}{\alpha}m, -c)$.

Let us now look at the second condition. A point (r,s) is above the line y = mx + c if s − rm > c or 0 > c + rm − s, or c + rm − s < 0.

The dual of point (r,s) is the line $y = \alpha(rx - \mu s)$, and the dual of the line y = mx + c is the point $(\mu m, -\alpha\mu c)$. Line $y = \alpha(rx - \mu s)$ is above the point $(\mu m, -\alpha\mu c)$, if $(-\alpha\mu c) - \alpha r(\mu m) < -\alpha\mu s$ or $-(\alpha\mu)(c + rm - s) < 0$. As $c + rm - s < 0, -(c + rm - s) > 0$, or $\alpha\mu < 0$.

Lemma 3: If $\alpha\mu < 0$, then if a point p lies above line L, then d(p) lies above d(L), and if $\alpha\mu > 0$, then if a point p lies above line L, then d(p) lies below d(L) (i.e. the order gets reversed).

Observation 2: Both conditions cannot be simultaneously satisfied.

If we define vertical distance [6] between a point (r,s) and the line y = mx + c to be |mr + c − s|, then, from the proof of lemma, in the dual space, the vertical distance gets scaled by $|\alpha\mu|$.

Parallel lines y = mx + c and y = mx + b will be mapped to points $(\mu m, -\alpha\mu c)$ and $(\mu m, -\alpha\mu b)$, or (vertical) distance between these two points is $|\alpha\mu(b - c)|$.

Thus, if we wish vertical distance to be preserved, then $\alpha\mu = \pm 1$.

Har-Peled [6], in Exercise 31.2, observes that no duality can preserve exactly orthogonal distances between points and lines.

Application

Assume S is a set of points in the plane. A point p is on the convex hull of S, if there is a non-vertical line L through p s.t., all points of S are below the line L.

For each point of S (including p), there is a line in dual space. Dual of line L is a point that lies on the line d(P). All dual lines d(S) should be on one side of d(L). Or d(L) lies on (upper or lower, depending on sign of $\alpha\mu$) envelope of lines d(S) [3,4,6].

Let us consider intersection of half-planes. Each half plane is either $y \leq m_i x + c_i$, which will be called top constraint or $y \geq m_i x + c_i$, which will be called bottom constraint.

The "feasible region" for top (respectively, bottom) constraints will be a convex region. Assume point (x_0,y_0) is on the boundary of this convex region. As it satisfies all top constraints, it is below lines $y = m_i x + c_i$ (if i is a top constraint). As it is on the boundary, it is on at least one such line. In dual space, the point (x_0,y_0) gets mapped to a line (say L_0), and each line $y = m_i x + c_i$ will get mapped to a point (say p_i). Moreover, all points p_i in the dual space will be on one side of the line L_0 and at least one point will be on this line. Thus, line L_0 will be tangent to the convex hull.

An edge e in the convex hull (in dual space) is a line between two dual points, say $d(L_1)$ and $d(L_2)$. This edge e will correspond to a point (say X) in the untransformed domain. As (dual) line d(X) (which contain segment e) passes through points $d(L_1)$ and $d(L_2)$, point X will lie on lines L_p and L_q, i.e., point X will be the point of intersection of these two lines.

Thus, line segments of convex hull define extreme point of feasible solution space, and both sets occur in the same relative order. Let us assume we have determined the convex hull.

We can similarly deal with bottom constraints B. As both these hulls are sorted (say on x coordinate), these can be merged in linear time [3,4,7]. The merged sets define a vertical slab, in which there are at most two boundary segments, one from the lower and one from the upper. Hence, we can determine the boundary of intersection of these half-planes in linear time.

Next some applications of intersection of half-planes are discussed [7]. In linear programming, assume we wish to maximise cx + dy. At each corner point (p_i,q_i) of solution space, we find $V_i = cp_i + dq_i$ and find the largest V_i. This again takes linear time, if the solution space is known. However, solution space is the intersection (common) region of half-planes.

Kernel of a polygon is the region from which entire polygon is visible. We interpret each edge of polygon as a half plane (the side containing the interior). Intersection of these regions will give the kernel.

Duality in d-dimensions

Let us generalise dual maps used (see e.g. [6,8]). Assume point $P = (p_1,\ldots, p_d)$ is mapped to hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ for some constants a_i, and hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point $(b_1m_1,b_2m_2,\ldots,b_{d-1}m_{d-1},b_dc)$, for constants b_i.

Edelsbrunner [8], on page 4, in the first map, takes all $a_i = 1$ and in the second, on page 13, $a_i = 2$ for $i = 1,\ldots,d-1$ and takes $a_d = -1$. In another map, on page 17, point P is mapped to the hyperplane $\sum_{i=1}^{d} p_i x_i = 1$. Har-Peled [6] considers the map $x_d = \sum_{i=1}^{d-1} p_i x_i + p_d$.

If the point P lies on the hyperplane L, then $p_d = \sum m_i p_i + c$. For the transformed point $(b_1m_1,\ldots,b_{d-1}m_{d-1},b_dc)$ to lie on transformed plane, we must have

$$(b_d c) = \sum a_i(b_i m_i)p_i + a_d p_d = \sum a_i(b_i m_i)p_i + a_d\left(\sum m_i p_i + c\right) = \sum m_i p_i(a_i b_i + a_d) + a_d c$$

. Thus, $b_d c = a_d c$ or $b_d = a_d$ and $a_i b_i = -a_d$ or $b_i = -a_i/a_d$.

Or the general transforms are as follows:

Point P = $(p_1,\dots, p_d)$ is mapped to hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ for some constants a_i.

Plane and hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point $\left(-a_d\frac{m_1}{a_1}, -a_d\frac{m_2}{a_2}, \dots, -a_d\frac{m_{d-1}}{a_{d-1}}, a_d c\right)$.

Dual of point $\left(-a_d\frac{m_1}{a_1}, -a_d\frac{m_2}{a_2}, \dots, -a_d\frac{m_{d-1}}{a_{d-1}}, a_d c\right)$ is the plane $x_d = \sum a_i\left(-a_d\frac{m_i}{a_i}\right)x_i + a_d(a_d c) = -\sum a_d m_i x_i + a_d^2 c$.

For this to be the original plane (dual to be its own inverse) $a_d = -1$. Thus, for this case, the transforms are:

Point P = $(p_1,\dots, p_d)$ is mapped to the hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i - p_d$ for some constants a_i.

Plane and hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point $\left(\frac{m_1}{a_1}, \frac{m_2}{a_2}, \dots, \frac{m_{d-1}}{a_{d-1}}, -c\right)$.

Let us now look at the second condition. A point P is above the plane L if $p_d \geq \sum m_i p_i + c$ or $p_d - \sum m_i p_i - c \geq 0$.

Dual of P is the hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$, and dual of hyperplane L is the point $\left(-a_d\frac{m_1}{a_1}, -a_d\frac{m_2}{a_2}, \dots, -a_d\frac{m_{d-1}}{a_{d-1}}, a_d c\right)$.

Dual of point (hyperplane) is above the dual of plane (point) if the dual point (of original hyper plane) is below dual plane (of original point); thus, $c a_d \leq \sum a_i p_i\left(-a_d\frac{m_i}{a_i}\right) + a_d p_d = a_d\left(p_d - \sum p_i m_i\right)$ or $0 \leq a_d\left(p_d - \sum p_i m_i - c\right)$. As the term in the brackets is positive, it implies a_d is also positive, i.e., $a_d > 0$. Thus, again, both conditions cannot be true.

Parallel hyperplanes $x_d = \sum m_i x_i + c$ and $x_d = \sum m_i x_i + b$ will get mapped to points $\left(-a_d\frac{m_1}{a_1}, -a_d\frac{m_2}{a_2}, \dots, -a_d\frac{m_{d-1}}{a_{d-1}}, a_d c\right)$ and $\left(-a_d\frac{m_1}{a_1}, -a_d\frac{m_2}{a_2}, \dots, -a_d\frac{m_{d-1}}{a_{d-1}}, a_d b\right)$. Thus, the vertical distance |c-b| gets scaled to |a_d(c-b)| to preserve vertical distance $a_d = \pm 1$ (as before).

Remark: As first d-1 coordinates of dual points are identical, other distances are not preserved.

Vertical distance between point $(p_1,\dots,p_d)$ and hyperplane $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is

$\left|p_d - \sum_{i=1}^{d-1} m_i x_i - c\right|$ and between dual hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ and dual point $-a_d(m_1/a_1, \ldots, m_{d-1}/a_{d-1}, -c)$ is $\left|a_d c + a_d \sum_{i=1}^{d-1} a_i p_i (m_i/a_i) - a_d p_d\right| = |a_d| \left|c + \sum_{i=1}^{d-1} p_i m_i - p_d\right|$. Thus, again distances will be preserved if $|a_d|=1$ or $a_d = \pm 1$.

Application to k-nearest neighbours and arrangements

An elementary and very intuitive treatment of relationship between arrangements in d+1 dimensions and searching for k-nearest neighbour in d-dimensions is next described.

Assume S = {P_1,..., P_n} is a set of n points in the R^d, the d-dimensional Euclidean space. Assume P = (p_1, ...,p_d) and Q = (q_1,...,q_d) are two points of S.

A query (or test) point X = (x_1,...,x_d) will be closer to P than to Q if

$$\sum_{i=1}^{d}(x_i - p_i)^2 < \sum_{i=1}^{d}(x_i - q_i)^2$$ or

$$\sum_{i=1}^{d}\left(x_i^2 - 2x_i p_i + p_i^2\right) < \sum_{i=1}^{d}\left(x_i^2 - 2x_i q_i + q_i^2\right)$$ or

$$2\sum_{i=1}^{d} x_i q_i - \sum_{i=1}^{d} q_i^2 < 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2.$$ For R = (r_1,...,r_d), define

$$f(R) = 2\sum_{i=1}^{d} x_i r_i - \sum_{i=1}^{d} r_i^2.$$

Then, above condition becomes f(P)>f(Q). As the equation $z = f(R) = 2\sum_{i=1}^{d} x_i r_i - \sum_{i=1}^{d} r_i^2$ is an equation of d+1 dimensional hyperplane, the above condition is equivalent to saying that hyperplane $z = 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2$ is above the hyperplane $z = 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2$.

Remark: Instead of z = f(R), we can also choose any monotonic function (like $z^2 = f(R)$, $\sqrt{z} = f(R)$, etc.). However, $z\# = f(R)$ appears to be the simplest choice.

Hence, if we draw an arrangement of n-hyperplanes [3,4,8] with plane z = f(P_i) for the ith point P_i, then the nearest neighbour of query point X will be the topmost hyperplane above X (the one visible from $(x_1, \ldots x_d, \infty)$). The second nearest neighbour will be the second topmost hyperplane and so on. Hence, to find the kth closest point, we need to only consider hyperplanes in the arrangement, which have (k-1) hyperplanes above them.

Edelsbrunner [8], on page 23, considers a dual map that transforms point P to $x_{d+1} = \sum_{i=1}^{d} 2p_i x_i - \sum_{i=1}^{d} p_i^2$. Here, a_i = 2 for i = 1, ..., d and $c = -\sum_{i=1}^{d} p_i^2$. Thus, the map $x_{d+1} \equiv z = f(P_i)$ can also be viewed as a dual mapping of a point in d-dimensions to a "special plane" (e.g., one constrained to be tangent to a paraboloid or a hyper-sphere) in d+1 dimensions.

Results

There are several notions of duality between lines and points. In this note, it is shown that all these can be generalised and studied in a unified way. Most interesting properties are also true for the generalised mapping. However, either a dual mapping can be its own inverse or it can preserve relative order (but not both).

Concept of duality between points and lines can be generalisation to higher dimensions also. An elementary and very intuitive treatment of relationship between arrangements of Computational Geometry in *d*+1 dimensions and searching for *k*-nearest neighbour in Euclidean space in *d*-dimensions is also given.

Discussion

Several notions of duality between points and lines in Computational Geometry can be generalised and unified. Many of the most interesting properties also hold under this generalised mapping. Therefore, by discussing or studying this generalised map, one can in single step study or understand all notions of duality proposed by various authors in computer science.

The mapping between kth level in arrangements in Computational Geometry in d+1 dimensions (which is useful for searching for k-nearest neighbour in d-dimensions) can also be viewed as a duality, which maps a d-dimensional point to a restricted hyperplane in d+1 dimensions.

Conclusions

There are several notions of duality between lines and points. All these can be studied in a unified way. It has been shown in this note that most properties of all dual maps are also true for the generalised map. However, either dual mapping can be its own inverse or it can preserve relative order (but not both).

Dual mappings can be generalised to d-dimensions. An elementary and very intuitive treatment of relationship between arrangements in Computational Geometry d+1 dimensions and searching for k-nearest neighbour in d-dimensions has also been described. This can also be viewed as duality that maps a d-dimensional point to a restricted hyperplane in d+1 dimensions.

Additional Information

Author Contributions

All authors have reviewed the final version to be published and agreed to be accountable for all aspects of the work.

Concept and design: Sanjeev Saxena

Drafting of the manuscript: Sanjeev Saxena

Critical review of the manuscript for important intellectual content: Sanjeev Saxena

Disclosures

Human subjects: All authors have confirmed that this study did not involve human participants or tissue. **Animal subjects:** All authors have confirmed that this study did not involve animal subjects or tissue. **Conflicts of interest:** In compliance with the ICMJE uniform disclosure form, all authors declare the following: **Payment/services info:** All authors have declared that no financial support was received from any organization for the submitted work. **Financial relationships:** All authors have declared that they have no financial relationships at present or within the previous three years with any organizations that might have an interest in the submitted work. **Intellectual property info:** Preprint posted at viXra:2502.0043 and submitted to arXiv (arXiv.org perpetual, non-exclusive license). **Other relationships:** This article was previously posted to the viXra preprint server (as viXra:2502.0043), and this article is not pending full publication elsewhere. Preprint has been posted on viXra full link: https://vixra.org/abs/2502.0043 (https://vixra.org/pdf/2502.0043v1.pdf and https://vixra.org/pdf/2502.0043v2.pdf). .

Acknowledgements

I wish to thank students who attended lectures of CS663 (2024-2025) for their comments and reactions.

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